

Stochastic Integrals of Point Processes and the Decomposition of Two-Parameter Martingales

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Let M be a square integrable martingale indexed by $[0, 1]^2$ with respect to a filtration which possesses the property of conditional independence. Assume that M has trajectories which are continuous for approach from the right upper quadrant and possess limits for the remaining three. M can have three kinds of jumps. A point t is a 0-jump if $\Delta_t M = \lim_{s \uparrow t} [M_t - M_{(t_1, s_2)} - M_{(s_1, t_2)} + M_s] \neq 0$, a 1-jump if $\Delta_t M = 0$ and $\lim_{s_1 \uparrow t_1} [M_t - M_{(s_1, t_2)}] \neq 0$. Analogously, 2-jumps are defined. With the 0-jumps associate the two-parameter point process μ^M which assigns unit point mass to nontrivial $(t, \Delta_t M)$, with the 1-jumps the one-parameter point process μ_1^M which puts unit mass to nontrivial $(t_1, \Delta_{t_1} M_{(\cdot, 1)})$, and with the 2-jumps a corresponding μ_2^M . We define stochastic integrals with respect to the compensated $\mu^M, \mu_i^M, i = 1, 2$, with the help of which we can describe the jump components associated with the respective jumps in the orthogonal decomposition of M by discontinuous and continuous parts. © 1989 Academic Press, Inc.

INTRODUCTION

A square integrable right continuous martingale possessing left limits admits an orthogonal decomposition into a unique pure jump part and a continuous component. The proof of this classical structure theorem for one-parameter martingales involves some deeper knowledge of the general theory of stochastic processes. For example, in order to obtain information on the continuity properties of the “compensators” of their jumps, it is useful to distinguish between previsible and totally inaccessible stopping times.

The absence of a similarly central notion of “stopping” slowed down the progress in founding a general theory of multi-parameter processes considerably, even under the simplifying assumption of conditional indepen-

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dence of the development of the underlying filtration in the different parameter directions. Accordingly, a structure theorem for two-parameter square integrable martingales proved to be essentially harder. Only recently, by considering "simple sets," i.e., random point sets or sets of horizontal/vertical line segments in parameter space, the number of which constitutes an integrable random variable, as the two-parameter substitutes of the graphs of stopping times, and by presenting satisfactory notions of previsibility and inaccessibility of sets of these types, a complete description of the orthogonal decomposition of a square integrable two-parameter martingale by three jump components and a continuous part could be given (see [9]). Simple sets partly generalize the notion of "stopping line" extensively studied, for example, by Merzbach [13, 14].

For a one-parameter martingale M , a direct description of the pure jump part can be given by means of a stochastic integral with respect to the compensated point process, which, to any non-trivial jump $\Delta_t M$ of M at t assigns unit point mass at $(t, \Delta_t M)$ in $\mathbb{R}_+ \times \mathbb{R}$ (see, for example, Jacod [10]). Several attempts have been made to follow this line of reasoning and describe the jump components of two-parameter martingales by integrals of point processes (see Mishura [17–20]). However, in lack of an adequate two-parameter substitute of previsible and inaccessible stopping times (see also Mazziotto, Merzbach, and Szpirglas [12]), the resulting decomposition theorem was only—roughly—for martingales without previsible jumps in either parameter direction.

In the present paper, we fill this gap and show that this restriction is completely unnecessary. We give a description of the three jump components of a square integrable two-parameter martingale M in terms of stochastic integrals of compensated point processes associated in a natural way with M . Doing so, on one hand, we obtain an alternative derivation of the decomposition theorem in [9, p. 156]. On the other hand, the stochastic integrals introduced for this purpose promise to be interesting in their own right. As is indicated by Itô's formula in the classical theory (see, for example, Jacod [10]) and by a first attempt to derive its counterpart for two-parameter martingales with jumps (see Mishura [20]), which covers part of the above-mentioned class of martingales and is hard to read, a forthcoming derivation of this central formula may reveal their significance.

To outline our program, we start with the famous regularity theorem of Bakry, Millet, and Sucheston [1, 16]. It states that any $L \log^+ L$ -integrable martingale M (i.e., $|M_1| \log^+ |M_1|$ is integrable) possesses a modification which has "regular" trajectories, i.e., continuous for approach from the right upper quadrant and provided with limits for approach from the remaining three quadrants. For regular martingales it makes sense to talk about jumps. We have to distinguish three kinds.

A point t in the parameter space $[0, 1]^2$ is a “0-jump” if $\Delta_t M = \lim_{s \uparrow t} [M_t - M_{(s_1, t_2)} - M_{(t_1, s_2)} + M_s] \neq 0$, and an “ i -jump” if $\Delta_t M = 0$ and $\lim_{s_i \uparrow t_i} [M_t - M_{(s_1, t_i)}] \neq 0$. Here \bar{i} is the complementary index of i and the coordinates of s, t are denoted by subscripts, $i = 1, 2$. To describe the 0-jump component M^0 of M , we consider the point process μ^M of 0-jumps, which associates unit point mass to any $(t, \Delta_t M)$ in $[0, 1]^2 \times \mathbb{R}$ for which the jump $\Delta_t M$ is nontrivial. A suitable extension of the dual projection theorem of Jacod [10] for random measures to the two-parameter situation allows to “compensate” μ^M by its previsible projections $(\mu^M)^{\pi_1}$, $(\mu^M)^{\pi_2}$, and $(\mu^M)^\pi$ in directions 1 (resp. 2, and 1 and 2). The fact that this compensation yields a “martingale measure” enables us to define the stochastic integral $W^\epsilon \mu^M$ for a class of previsible processes W on $\Omega \times [0, 1]^2 \times \mathbb{R}$ with respect to this measure. M^0 is identified with $W_0^\epsilon \mu^M$, where $W_0(\cdot, \cdot, x) = x$, $x \in \mathbb{R}$. Now $M - M^0$ has at most 1- and 2-jumps. The description of the 1- and 2-jump components of this martingale turns out to be particularly simple. To obtain the former, for example, we consider the jumps of the one-parameter martingale $(M - M^0)_{(\cdot, 1)}$. Its pure jump part is given as usual by the stochastic integral of the process $W_1(\cdot, \cdot, x) = x$, $x \in \mathbb{R}$, on $\Omega \times [0, 1] \times \mathbb{R}$ with respect to the compensated random measure μ_1^M , which assigns unit point mass to any point $(t, \Delta_t (M - M^0)_{(\cdot, 1)})$ in $[0, 1] \times \mathbb{R}$ for which $\Delta_t (M - M^0)_{(\cdot, 1)} \neq 0$. Now the crucial observation is that in consequence of the conditional independence of the development in the two-parameter directions, the processes $t_2 \rightarrow \Delta_{t_1} (M - M^0)_{(\cdot, t_2)}$ are martingales, $t = (t_1, t_2)$. Therefore, the optional projection of $W_1^\epsilon \mu_1^M$ in direction 2 must yield the 1-jump part of M . A similar construction gives the 2-jump component.

1. NOTATIONS, DEFINITIONS, AND BASICS

The stochastic processes considered in this paper are parametrized by $I = [0, 1]$ or $\mathbb{I} = [0, 1]^2$. The latter interval is ordered by “ \leq ” which is understood to be the coordinatewise linear ordering of I . Intervals with respect to this ordering are defined as usual. If J is an interval, we write s^J, t^J for its respective lower and upper corners. By a partition of a parameter interval we always mean a partition generated by a finite number of axial parallel lines (points) consisting of left open, right closed intervals (in the relative topology of $\mathbb{I}(I)$). A 0-sequence of partitions is a sequence of partitions which is increasing with respect to fineness and the mesh of which goes to 0. To denote components of points in \mathbb{I} , we use lower indices. For example, $t = (t_1, t_2)$ for $t \in \mathbb{I}$. We sometimes write $t = (t_i, t_{\bar{i}})$ regardless of whether $i = 1$ or 2, where \bar{i} denotes the complementary index $3 - i$ of i . Given a function $f: I \rightarrow \mathbb{R}$, the increment of f over an

interval J in I will be written $\Delta_J f$. This also applies to functions $f: \mathbb{I} \rightarrow \mathbb{R}$. Here $\Delta_J f = f(t') - f(s'_1, t'_2) - f(t'_1, s'_2) + f(s')$. f is called increasing if $\Delta_J f \geq 0$ for all intervals J , regular, if

$$\lim_{s \downarrow t} f(s) = f(t), \quad \lim_{s \uparrow t} f(s), \quad \lim_{s_1 \uparrow t_1, s_2 \downarrow t_2} f(s), \quad \lim_{s_1 \downarrow t_1, s_2 \uparrow t_2} f(s)$$

exist, for $t \in \mathbb{I}$. If \mathfrak{A} is a system of sets, \mathfrak{A}^σ abbreviates the system of all countable unions of sets in \mathfrak{A} and $|\mathfrak{A}|$ is the cardinality of \mathfrak{A} .

Given measurable spaces (X, \mathfrak{A}) , (Y, \mathfrak{B}) , the set of $\mathfrak{A} - \mathfrak{B}$ -measurable (nonnegative) functions mapping X to Y is denoted by $\mathfrak{M}(\mathfrak{A}, \mathfrak{B})$ ($\mathfrak{M}^+(\mathfrak{A}, \mathfrak{B})$). For the Borel sets of a topological space X the symbol $\mathfrak{B}(X)$ is used. Our basic probability space is $(\Omega, \mathfrak{F}, P)$. \mathfrak{F} is assumed to be complete with respect to P . The filtration $\mathbb{F} = (\mathfrak{F}_t)_{t \in \mathbb{I}}$ which is fixed throughout the paper is supposed to satisfy some basic assumptions: it is right continuous, i.e., $\mathfrak{F}_t = \bigcap_{s > t} \mathfrak{F}_s$ for all $t \in \mathbb{I}$, it is complete, i.e., \mathfrak{F}_t contains all P -zero sets, and, for convenience, \mathfrak{F}_t is trivial whenever $t \in \mathbb{I} \cap \partial \mathbb{R}_+^2$. The most important hypothesis, however, is the "conditional independence" of the filtrations $\mathbb{F}_1 = (\mathfrak{F}_{t_1}^1)_{t_1 \in I}$ and $\mathbb{F}_2 = (\mathfrak{F}_{t_2}^2)_{t_2 \in I}$, where $\mathfrak{F}_{t_i}^i = \mathfrak{F}_{(t_i, 1)}$, $i = 1, 2$. It states that for all $t \in \mathbb{I}$, the σ -algebras $\mathfrak{F}_{t_1}^1$ and $\mathfrak{F}_{t_2}^2$ are conditionally independent given \mathfrak{F}_t , and is often referred to as the (F4)-condition of Cairoli and Walsh [5]. The evanescent sets in $\Omega \times \mathbb{I}$ ($\Omega \times I$) are those sets whose Ω -projections are P -zero. They are denoted by \mathfrak{N} in both cases. Stochastic processes are a priori no more than mere families of random variables. A stochastic process X on $\Omega \times \mathbb{I}$ defines two families of one-parameter processes: for $t_i \in I$, $X_{(\cdot, t_i)}$ is the process $(\omega, t_i) \rightarrow X_t(\omega)$, $i = 1, 2$. Two processes X and Y are considered as being equal, if they differ on an evanescent set, as being versions of each other, if $X_t = Y_t$, a.s. for all t . A process X is called increasing (regular), if for all $\omega \in \Omega$ the trajectories $X(\omega, \cdot)$ are increasing (regular). Besides the usual Banach spaces of random variables $L^p(\Omega, \mathfrak{F}, P)$ with norm $\|\cdot\|_p$, we will eventually have to consider the following Banach spaces of processes. $L^{p, \infty}(\Omega \times \mathbb{I}, \mathfrak{F} \times \mathfrak{B}(\mathbb{I}), P)$ ($L^{p, \infty}(\Omega \times I, \mathfrak{F} \times \mathfrak{B}(I), P)$) is the vector space of all processes which are $\mathfrak{F} \times \mathfrak{B}(\mathbb{I})$ - ($\mathfrak{F} \times \mathfrak{B}(I)$ -) measurable and for which $\|X\|_{p, \infty} = \sup_{t \in \mathbb{I}(I)} \|X_t\|_p < \infty$, topologized by these functionals, $p \geq 1$.

By far the most important measurability concepts for stochastic processes are evoked by the words "optionality" and "previsibility." We will briefly recall the definitions relevant for us. A process X on $\Omega \times I$ is called \mathbb{F}_i -adapted if $X_t \in \mathfrak{M}(\mathfrak{F}_t^i, \mathfrak{B}(\mathbb{R}))$ for $t \in I$. The σ -algebra \mathfrak{G}_i (\mathfrak{P}_i) of \mathbb{F}_i -optional (\mathbb{F}_i -previsible) sets is generated by the right continuous (continuous) \mathbb{F}_i -adapted processes on $\Omega \times I$ possessing left limits. For processes on $\Omega \times \mathbb{I}$, the following are important: $\mathfrak{G}^i = [\mathfrak{G}_i \times \mathfrak{B}(I)] \vee \mathfrak{N}$ (resp. $\mathfrak{P}^i = [\mathfrak{P}_i \times \mathfrak{B}(I)] \vee \mathfrak{N}$) are the σ -algebras of i -optional (resp. i -previsible) sets, $i = 1, 2$. A process X on $\Omega \times \mathbb{I}$ is called adapted if $X_t \in \mathfrak{M}(\mathfrak{F}_t, \mathfrak{B}(\mathbb{R}))$ for

$t \in \mathbb{I}$. The σ -algebras \mathfrak{G} of optional (\mathfrak{P} of previsible) sets are generated by the adapted regular (continuous) processes. In consequence of conditional independence, we have the important equations $\mathfrak{G} = \mathfrak{G}^1 \cap \mathfrak{G}^2$, $\mathfrak{P} = \mathfrak{P}^1 \cap \mathfrak{P}^2$ (see [9, p. 89]). For the definitions and basic properties of i -optional, optional (i -previsible, previsible) projections of bounded $[\mathfrak{F} \times \mathfrak{P}(\mathbb{I})] \vee \mathfrak{M}$ -measurable processes X , denoted respectively by iX , iX (iX , iX) and of their dual counterparts for integrable increasing processes A , denoted respectively by A^i , A^i (A^i , A^i), $i=1, 2$, the reader is referred to [9]. Note that the projections can be extended to random variables X , considered as processes with a trivial t -dependence, satisfying $E(|X| \log^+ |X|) < \infty$ (see Bakry [3]). (Dual) optional and previsible projections of processes (resp. increasing processes) of one parameter with respect to the filtration \mathbb{F}_i are also denoted by the superscripts γ_i , γ , π_i , π and called (dual) i -optional and i -previsible projections.

To analyze the jumps of processes, the following concepts of “thin” optional sets will be of vital interest. A set $T \subset \Omega \times I$ is said to be \mathbb{F}_i -simple if $T \in \mathfrak{G}_i$ and $\omega \rightarrow |T_\omega|$ is integrable. If it is clear from the context what i is, the suffix “ \mathbb{F}_i ” may be omitted. In $\Omega \times \mathbb{I}$, the geometry of simple sets is richer. A set $T \in \mathfrak{G}$ is called 0-simple, if $\omega \rightarrow |T_\omega|$ is integrable, 1-simple, if T_ω consists of finitely many vertical open line segments whose upper boundary is on $\partial \mathbb{I}$ for $\omega \in \Omega$, the number of which constitutes an integrable random variable, 2-simple, if an analogous statement for horizontal line segments can be made. By \mathcal{S}^i we denote the system of i -simple sets, by $s(\mathcal{S}^i)$ the semiring generated by \mathcal{S}^i , $i=0, 1, 2$. Note that $\mathcal{S}^0 = s(\mathcal{S}^0)$. \mathbb{F}_i -simple sets (0-simple sets) T are sometimes studied by means of their associated increasing process $\Gamma(T)$, defined by $\Gamma(T)_t(\omega) = |T_\omega \cap [0, t]|$, $(\omega, t) \in \Omega \times I$ ($\Omega \times \mathbb{I}$). $\Gamma(T)_t$ just counts the number of points in T up to t . It has been shown in [9] that in analogy to the graphs of stopping times in the classical theory, simple sets can be decomposed by simple sets of different “accessibility” properties. An \mathbb{F}_i -simple set $T \subset \Omega \times I$ is said to be \mathbb{F}_i -inaccessible if for any \mathbb{F}_i -simple $S \in \mathfrak{P}_i$ the intersection $S \cap T$ is evanescent. A 0-simple set $T \subset \Omega \times \mathbb{I}$ is called i -previsible, i -inaccessible (totally inaccessible) if, for any 0-simple $S \in \mathfrak{P}$ ($S \in \mathfrak{P}^1 \cup \mathfrak{P}^2$), the intersection $S \cap T$ is evanescent. Similarly a set $T \in s(\mathcal{S}^i)$ is said to be inaccessible if the intersection with any previsible $S \in \mathcal{S}^i$ is evanescent, $i=1, 2$. Theorems on the decomposition of simple sets in $\Omega \times \mathbb{I}$ by inaccessible/previsible simple sets are presented in [9]. They extend in an obvious way to simple sets in $\Omega \times I$ and are, of course, covered by the classical decomposition theorem, where graphs of inaccessible/previsible stopping times take their part (see Dellacherie and Meyer [6]).

The most important class of processes we will have to discuss here are the martingales. An integrable, \mathbb{F}_i -adapted process on $\Omega \times I$ is called \mathbb{F}_i -martingale if it is a martingale with respect to \mathbb{F}_i . An integrable, adapted

process M on $\Omega \times \mathbb{I}$ is called martingale if for $s, t \in \mathbb{I}$, $s \leq t$, we have $E(M_t | \mathfrak{F}_s) = M_s$. Due to the conditional independence property, M is a martingale iff $M_{(\cdot, t_i)}$ is an \mathbb{F}_i -martingale for any $t_i \in I$, $i = 1, 2$. According to the famous regularity theorem of Bakry, Millet, and Sucheston [1, 16] any $L \log^+ L$ -integrable martingale M (i.e., $E(|M_1| \log^+ |M_1|) < \infty$) possesses a version with regular trajectories. For regular martingales, the following three kinds of jumps are well defined and will prove to be relevant. A point $(\omega, t) \in \Omega \times \mathbb{I}$ is called 0-jump, if $\Delta_t M(\omega) = \lim_{s \uparrow t} \Delta_{[s, t]} M(\omega) \neq 0$ and i -jump, if $\Delta_t M(\omega) = 0$ and $\Delta_{t_i} M_{(\cdot, t_i)}(\omega) = \lim_{s_i \uparrow t_i} \Delta_{[s_i, t_i]} M_{(\cdot, t_i)}(\omega) \neq 0$, $i = 1, 2$. It is shown in [9, pp. 120–123], that the set of discontinuities of a regular martingale is contained in a countable union of simple sets. Analogously, jumps of one-parameter right continuous martingales possessing left limits are treated. The Hilbert space of regular square integrable martingales, the topology of which is defined by $\|\cdot\|_2$, is denoted by \mathcal{M}^2 . We finally emphasize that, for convenience of notation, all martingales to be considered in this paper are assumed to vanish on $\mathbb{I} \cap \partial \mathbb{R}_+^2$.

2. PROJECTION AND DECOMPOSITION OF RANDOM MEASURES

This section is devoted to the presentation of some preparatory material, some of which, however, may be interesting in its own right. In the first part the problem of defining (dual) projections of given random measures will be considered. Since our parameter space is 2-dimensional, a closer description of the most important one of these problems involves projections in each one of the two parameter directions separately, the other parameter being fixed, as well as projections in both directions simultaneously. As in the classical theory, the Doléans measure provides one possible key to its solution. In Theorem 1 we employ it to obtain a useful criterion for the existence of optional (previsible) random measures which are candidates for the respective projections. Given a random measure, in Theorem 2 we show the existence and uniqueness of its optional (previsible) projections in the two-parameter directions separately as well as simultaneously. The property of conditional independence of \mathbb{F}_1 and \mathbb{F}_2 implies that the simultaneous projections are merely the products of the separate projections in an arbitrary order. Our presentation follows Jacod's [10] book closely. In the second part of this section, we consider a random measure and show how it can be decomposed into three "jump parts" and a "continuous part." The results of Theorem 3 also include statements on the inheritance of optionality and previsibility properties of the given random measure by its components. We conclude this section by a brief investigation of a particularly well-behaved class of random

measures, which, in Section 2 will be seen to contain all those we need for our applications. They can be characterized by the statement that they are σ -finite and at each point in parameter space they can gain at most finite mass. In Theorem 4 they turn out to possess the nice property of living on countable unions of simple sets. For the sake of completeness and for later reference we will, along with the two-parameter random measures we are primarily interested in, describe the corresponding (well-known) properties of their one-parameter counterparts.

We first introduce the notion of random measures and fix some important notations concerning their properties and integrability. Throughout this section we denote by H an arbitrary Lusin space, i.e., the homeomorphic image of a Borel subset of a compact metric space.

DEFINITION 1. Let $J = \mathbb{I}$, I . A “random measure on J ” is a function

$$\mu: \Omega \times \mathfrak{B}(J) \times \mathfrak{B}(H) \rightarrow \mathbb{R}_+ \cup \{\infty\}$$

such that

- (a) $\mu(\omega, \cdot)$ is a σ -finite measure on $\mathfrak{B}(J) \times \mathfrak{B}(H)$ for all $\omega \in \Omega$,
- (b) $\mu(\cdot, A)$ is \mathfrak{F} -measurable for all $A \in \mathfrak{B}(J) \times \mathfrak{B}(H)$.

DEFINITION 2. Let μ be a random measure. Then $P \times \mu$ is called “Doléans measure” of μ .

If μ is a random measure on $J = \mathbb{I}$, I , we denote by $L^1(\mu)$ (resp. $L^1(P \times \mu)$; resp. $L^1(P \times \mu)^0$) the vector space of all processes $W \in \mathfrak{M}(\mathfrak{F} \times \mathfrak{B}(J) \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$ such that $W(\omega, \cdot)$ is integrable w.r.t. $\mu(\omega, \cdot)$ for P -a.e. $\omega \in \Omega$ (resp. W is integrable w.r.t. $P \times \mu$; resp. $W(\cdot, t, \cdot)$ is integrable w.r.t. $P \times \mu(\cdot, \{t\} \times \cdot)$) for all $t \in J$. Trivially, $L^1(P \times \mu) \subset L^1(P \times \mu)^0$. Fubini’s theorem implies that $L^1(P \times \mu) \subset L^1(\mu)$. For $W \in L^1(\mu)$ (resp. $\mathfrak{M}^+(\mathfrak{F} \times \mathfrak{B}(J) \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$), $t \in J$, we define the integral process

$$W \cdot \mu_t = \int_{[0, t] \times H} W(\cdot, s, x) \mu(\cdot, ds, dx).$$

For finite $W \in \mathfrak{M}^+(\mathfrak{F} \times \mathfrak{B}(J) \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$, $W\mu$ denotes the random measure defined by the integral of W w.r.t. μ . For example, given $A \in \mathfrak{F} \times \mathfrak{B}(J) \times \mathfrak{B}(H)$, $1_A \mu$ is μ , restricted to A .

According to the following definition, optionality and previsibility properties of random measures μ are simply properties of their integral processes $W \cdot \mu$.

DEFINITION 3. 1. Let μ be a random measure on I , $i = 1, 2$. μ is called " \mathbb{F}_i -optional" (" \mathbb{F}_i -previsible") if for every $W \in \mathfrak{M}^+(\mathbb{G}_i \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$ ($\mathfrak{M}^+(\mathfrak{P}_i \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$) the process $W \cdot \mu$ is \mathbb{G}_i -(\mathfrak{P}_i -) measurable.

2. Let μ be a random measure on \mathbb{I} , $i = 1, 2$. μ is called " i -optional" (" i -previsible") if for every $W \in \mathfrak{M}^+(\mathbb{G}^i \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$ ($\mathfrak{M}^+(\mathfrak{P}^i \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$) the process $W \cdot \mu$ is \mathbb{G}^i -(\mathfrak{P}^i -) measurable. By a corresponding statement, with \mathbb{G} (\mathfrak{P}) instead of \mathbb{G}^i (\mathfrak{P}^i), " $optional$ " (" $previsible$ ") random measures on \mathbb{I} are defined.

Since by conditional independence $\mathbb{G} = \mathbb{G}^1 \cap \mathbb{G}^2$ and $\mathfrak{P} = \mathfrak{P}^1 \cap \mathfrak{P}^2$ (see [9, p. 89]), the following proposition is obvious.

PROPOSITION 1. Let μ be a random measure on \mathbb{I} . Then μ is optional (previsible) iff μ is 1- and 2-optional (1- and 2-previsible).

If $1 \in L^1(P \times \mu)$, the mass that μ assigns to $\mathbb{I} \times H$ (resp. $I \times H$) is finite, P -a.s. In most of the cases, however, only one of the following " σ -integrability" properties holds.

DEFINITION 4. Let μ be a random measure on I , $i = 1, 2$.

(a) μ is said to be " $integrable$ " if $1 \in L^1(P \times \mu)$,

(b) μ is called " \mathbb{F}_i -optionally σ -integrable" (" \mathbb{F}_i -previsibly σ -integrable"), if there is a sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathbb{G}_i \times \mathfrak{B}(H)$ ($\mathfrak{P}_i \times \mathfrak{B}(H)$) satisfying $A_n \uparrow \Omega \times I \times H$ such that $1_{A_n} \mu$ is integrable for all $n \in \mathbb{N}$.

Similarly, the notions " $integrable$," " i -optionally σ -integrable," " i -previsibly σ -integrable," " $optionally \sigma$ -integrable," and " $previsibly \sigma$ -integrable" are defined for random measures on \mathbb{I} .

If μ is an i -optional random measure, the adverb " i -optionally" in statements of the property " i -optionally σ -integrable" may be suppressed, etc.

Remark. If μ is a random measure on $J = \mathbb{I}$, I , $W \in \mathfrak{M}^+(\mathfrak{F} \times \mathfrak{B}(J) \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$, obviously $W\mu$ is integrable iff $W \in L^1(P \times \mu)$. If μ is σ -integrable, $P \times \mu$ is σ -finite.

In the following theorem, a necessary and sufficient condition for a measure on $\mathfrak{F} \times \mathfrak{B}(J) \times \mathfrak{B}(H)$ to be the Doléans measure of a random measure on J is given, $J = \mathbb{I}, I$. In addition, it presents criteria for the optionality (previsibility) of the respective random measures.

THEOREM 1. Let m be a measure on $\mathfrak{F} \times \mathfrak{B}(\mathbb{I}) \times \mathfrak{B}(H)$, $i = 1, 2$. There exists an i -optional σ -integrable (i -previsibly σ -integrable) random measure μ such that $P \times \mu = m$ if and only if the following conditions are satisfied:

- (a) *there exists a sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{G}^i \times \mathfrak{B}(H)$ ($\mathfrak{P}^i \times \mathfrak{B}(H)$) such that $A_n \uparrow \Omega \times \mathbb{I} \times H$ and $1_{A_n} m$ is finite for $n \in \mathbb{N}$,*
- (b) *$m(N \times H) = 0$ for evanescent N ,*
- (c) *for any $A \in \mathfrak{G}^i \times \mathfrak{B}(H)$ ($\mathfrak{P}^i \times \mathfrak{B}(H)$) such that $1_A m$ is finite and any bounded $X \in \mathfrak{M}([\mathfrak{F} \times \mathfrak{B}(\mathbb{I})] \vee \mathfrak{N}, \mathfrak{B}(\mathbb{R}))$ we have*

$$\int X 1_A dm = \int \gamma_i X 1_A dm \left(\int \pi_i X 1_A dm \right).$$

In this case, μ is uniquely determined.

A similar statement holds for \mathfrak{G} (\mathfrak{P}) instead of \mathfrak{G}^i (\mathfrak{P}^i) and for random measures on I .

Proof. For the i -optional and i -previsible case, the proof is the same as in Jacod [10, pp. 70–72]. Let us concentrate on the optional case, the arguments for the previsible one being identical. Denote by (S_i) (resp. (S)) the statement (c) for \mathfrak{G}^i (resp. \mathfrak{G}), $i = 1, 2$. We will show that (S) is equivalent to (S_1) and (S_2) . If (S_1) and (S_2) are fulfilled, (S) follows from the fact that for $X \in \mathfrak{M}([\mathfrak{F} \times \mathfrak{B}(\mathbb{I})] \vee \mathfrak{N}, \mathfrak{B}(\mathbb{R}))$ bounded we have $\gamma_1 \gamma_2 X = \gamma_2 \gamma_1 X = \gamma X$ (see [9, p. 89]). Now assume that (S) holds. Let $T \in \mathfrak{G}^1$ and $G \in \mathfrak{B}(H)$ and suppose that $A = T \times G$ is such that $1_A m$ is finite. Then by (S) for any bounded $X \in \mathfrak{M}([\mathfrak{F} \times \mathfrak{B}(\mathbb{I})] \vee \mathfrak{N}, \mathfrak{B}(\mathbb{R}))$,

$$\begin{aligned} \int X 1_A dm &= \int X 1_T 1_G dm = \int \gamma(X 1_T) 1_G dm \\ &= \int \gamma(\gamma_1 X 1_T) 1_G dm = \int \gamma_1 X 1_A dm. \end{aligned}$$

A monotone class argument yields (S_1) . Similarly, (S_2) follows from (S) , and we have established the equivalence of (S) with (S_1) and (S_2) . Now the theorem has already been verified in the 1-optional and 2-optional cases. We therefore can conclude that there exists a 1-optional and 2-optional σ -integrable random measure μ such that $P \times \mu = m$ iff (a), (b), (c) for \mathfrak{G} are satisfied. An appeal to Proposition 1 completes the proof. ■

COROLLARY. *Let $i = 1, 2$, μ and ν be i -optional (i -previsible) σ -integrable random measures on \mathbb{I} such that $P \times \mu$ and $P \times \nu$ agree on $\mathfrak{G}^i \times \mathfrak{B}(H)$ ($\mathfrak{P}^i \times \mathfrak{B}(H)$). Then $\mu = \nu$. A similar statement holds for \mathfrak{G} (\mathfrak{P}) instead of \mathfrak{G}^i (\mathfrak{P}^i) and for random measures on I .*

The following theorem on optional and previsible projections of random measures is an easy consequence of Theorem 1.

THEOREM 2. Let $i=1, 2$, μ an i -optionally (i -previsibly) σ -integrable random measure on \mathbb{I} . Then there exists a unique i -optional (i -previsible) σ -integrable random measure μ^{γ_i} (μ^{π_i}) on \mathbb{I} such that the following equivalent statements hold:

(a) the measures $P \times \mu$ and $P \times \mu^{\gamma_i}$ ($P \times \mu^{\pi_i}$) agree on $\mathfrak{G}^i \times \mathfrak{B}(H)$ ($\mathfrak{P}^i \times \mathfrak{B}(H)$),

(b) for any $W \in L^1(P \times \mu)$ which is $\mathfrak{G}^i \times \mathfrak{B}(H)$ - ($\mathfrak{P}^i \times \mathfrak{B}(H)$ -) measurable

$$(W \cdot \mu)^{\gamma_i} = W \cdot \mu^{\gamma_i} \quad ((W \cdot \mu)^{\pi_i} = W \cdot \mu^{\pi_i}).$$

An analogous statement holds for \mathfrak{G} (\mathfrak{P}) instead of \mathfrak{G}^i (\mathfrak{P}^i) and for random measures on I . Moreover, $\mu^{\gamma_1 \gamma_2} = \mu^{\gamma_2 \gamma_1} = \mu^\gamma$ and $\mu^{\pi_1 \pi_2} = \mu^{\pi_2 \pi_1} = \mu^\pi$.

Proof. The existence and uniqueness of μ^{γ_i} (μ^{π_i}) as well as the equivalence of (a) and (b) follow from Theorem 1 as in Jacod [10, p. 73]. The remainder of the assertion is an immediate consequence of uniqueness, (b), and [9, p. 89]. ■

DEFINITION 5. Let μ be a random measure on \mathbb{I} which is i -optionally (i -previsibly) σ -integrable. The random measure μ^{γ_i} (μ^{π_i}) according to Theorem 2 is called " i -optional" (" i -previsible") projection" of μ . Correspondingly, the "optional projection" μ^γ and the "previsible projection" μ^π of a random measure μ on \mathbb{I} as well as the " \mathbb{F}_i -optional projection" μ^{γ_i} and the " \mathbb{F}_i -previsible projection" μ^{π_i} of a random measure μ on I are defined, $i=1, 2$.

According to the following proposition, moments of integral processes of the projections of random measures are bounded by the corresponding integral processes of the random measures themselves.

PROPOSITION 2. 1. Let $i=1, 2$, μ an i -optionally (i -previsibly) σ -integrable random measure on \mathbb{I} , W a nonnegative $\mathfrak{G}^i \times \mathfrak{B}(H)$ - ($\mathfrak{P}^i \times \mathfrak{B}(H)$ -) measurable process in $L^1(P \times \mu)$, $p \geq 1$. Then respectively

$$\|W \cdot \mu_1^{\alpha_i}\|_p \leq p \|W \cdot \mu_1\|_p, \quad \alpha = \gamma, \pi.$$

A similar statement holds for random measures on I .

2. Let μ be an optionally (i -previsibly) σ -integrable random measure on \mathbb{I} , W a nonnegative $\mathfrak{G} \times \mathfrak{B}(H)$ - ($\mathfrak{P} \times \mathfrak{B}(H)$ -) measurable process in $L^1(P \times \mu)$, $p \geq 1$. Then respectively

$$\|W \cdot \mu_1^\alpha\|_p \leq p^2 \|W \cdot \mu_1\|_p, \quad \alpha = \gamma, \pi.$$

Proof. By (b) of Theorem 2, $W \cdot \mu$ is an integrable increasing process on \mathbb{I} satisfying

$$(W \cdot \mu)^{\gamma_i} = W \cdot \mu^{\gamma_i} \quad ((W \cdot \mu)^{\pi_i} = W \cdot \mu^{\pi_i}).$$

Therefore, 1 follows from [9, p. 51]. Similarly, 2 follows from [9, p. 91]. ■

For the rest of this section we will be concerned with the problem of decomposing a given random measure into components which correspond to its three kinds of “discontinuities,” the “point jumps,” and the “axial jumps” in directions 1 and 2, and a “continuous” component, which are pairwise orthogonal. To begin with, we consider the point jumps. As far as random measures on \mathbb{I} are concerned, we concentrate on optional ones from now on.

PROPOSITION 3. *Let μ be an optional σ -integrable random measure on \mathbb{I} ,*

$$D^0(\mu) = \{(\omega, t) \in \Omega \times \mathbb{I} : \mu(\omega, \{t\} \times H) > 0\}.$$

Then $D^0(\mu) \in \mathfrak{G}$ and $\mu^0 = 1_{D^0(\mu) \times H} \mu$ is an optional σ -integrable random measure on \mathbb{I} . A corresponding statement holds for random measures on I . If, in addition, μ is i -previsible (previsible) σ -integrable, $D^0(\mu) \in \mathfrak{P}^i(\mathfrak{P})$ and μ^0 is i -previsible (previsible) σ -integrable, $i = 1, 2$.

Proof. The arguments for the “previsible part” of the assertion being identical, we concentrate on the “optional part.” Let $(A_n)_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{G} \times \mathfrak{B}(H)$ such that $A_n \uparrow \Omega \times \mathbb{I} \times H$ and $1_{A_n} \mu$ is integrable for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ set

$$C_n = 1_{A_n} \cdot \mu, \quad T_n = \{(\omega, t) \in \Omega \times \mathbb{I} : \Delta_t C_n(\omega) > 1/n\}.$$

Since C_n is an optional integrable increasing process, T_n is an optional set (see [9], p. 37). Moreover, $T_n \uparrow D^0(\mu)$. Hence $D^0(\mu) \in \mathfrak{G}$ and therefore $\mu^0 = 1_{D^0(\mu) \times H} \mu$ is optional. Since $\mu^0 \leq \mu$, μ^0 is optionally σ -integrable. ■

Let us next assume that μ has no point jumps. Then we can define axial jump components according to the following proposition.

PROPOSITION 4. *Let μ be an optional σ -integrable random measure on \mathbb{I} which satisfies $\mu(\omega, \{t\} \times H) = 0$ for $(\omega, t) \in \Omega \times \mathbb{I}$. Let*

$$D^j(\mu) = \{(\omega, t) \in \Omega \times \mathbb{I} : \mu(\omega, \{t_j\} \times [0, t_j] \times H) > 0\}, \quad j = 1, 2.$$

Then $D^j(\mu) \in \mathfrak{G}$ and $\mu^j = 1_{D^j(\mu) \times H} \mu$ is an optional σ -integrable random measure on \mathbb{I} . Moreover, $D^j(\mu) \in \mathfrak{P}^j$ and μ^j is j -previsible. If in addition, μ is i -previsible (previsible) σ -integrable, $D^j(\mu) \in \mathfrak{P}^i(\mathfrak{P})$ and μ^j is i -previsible (previsible) σ -integrable, $i, j = 1, 2$.

Proof. Again, we only argue for the optional case. Let $(A_n)_{n \in \mathbb{N}}$ and $(C_n)_{n \in \mathbb{N}}$ be as in the preceding proof and set

$$T_n^j = \{(\omega, t) \in \Omega \times \mathbb{I} : \Delta_{t_j}(C_n)_{(\cdot, t_j)}(\omega) > 1/n\}, \quad n \in \mathbb{N}, \quad j = 1, 2.$$

Now the process $(\omega, t) \rightarrow \Delta_{t_j}(C_n)_{(\cdot, t_j)}(\omega)$ is optional and, by hypothesis, continuous in direction j , thus even \mathfrak{P}^j -measurable (cf. [9, p. 37]). Hence as before $D^j(\mu) \in \mathfrak{G}$, μ^j is optional σ -integrable and, moreover, $D^j(\mu) \in \mathfrak{P}^j$. To see that μ^j is j -previsible, let $W \in \mathfrak{M}^+(\mathfrak{P}^j \times \mathfrak{B}(H), \mathfrak{B}(\mathbb{R}))$ be bounded. Then for any $n \in \mathbb{N}$

$$W \cdot (1_{T_n^j \times H \cap A_n} \mu)^j = W 1_{T_n^j \times H \cap A_n} \cdot \mu^j$$

is an optional integrable increasing process which is continuous in j -direction, hence j -previsible. Now monotone convergence gives the j -previsibility of $W \cdot \mu^j$, consequently of μ^j . This completes the proof. ■

Combining Propositions 3 and 4, we obtain the main result on the decomposition of random measures.

THEOREM 3. 1. *Let μ be an optional σ -integrable random measure on \mathbb{I} . Then there exist optional σ -integrable random measures $\mu^0, \mu^1, \mu^2, \mu^c$ such that*

$$(a) \quad \mu^0 = 1_{D^0(\mu) \times H} \mu, \text{ where}$$

$$D^0(\mu) = \{(\omega, t) \in \Omega \times \mathbb{I} : \mu(\omega, \{t\} \times H) > 0\} \in \mathfrak{G},$$

$$(b) \quad \mu^j = 1_{D^j(\mu') \times H} \mu, \text{ where}$$

$$D^j(\mu') = \{(\omega, t) \in \Omega \times \mathbb{I} : \mu'(\omega, \{t_j\} \times [0, t_j] \times H) > 0\} \in \mathfrak{G}, \quad j = 1, 2,$$

$$(c) \quad \mu^c = 1_{\overline{D^0(\mu) \cup D^1(\mu') \cup D^2(\mu')}} \times H \mu, \text{ for } \mu' = 1_{\overline{D^0(\mu)}} \times H \mu,$$

$$(d) \quad \mu = \mu^0 + \mu^1 + \mu^2 + \mu^c.$$

Moreover,

$$(e) \quad \mu^j \text{ and } D^j(\mu) \text{ are } j\text{-previsible, } j = 1, 2, \mu^c \text{ is previsible.}$$

If in addition, μ is i -previsible (previsible) σ -integrable, the same holds true for its four components, and $D^0(\mu), D^j(\mu') \in \mathfrak{P}^i(\mathfrak{P}), i = 1, 2$.

2. Let $i = 1, 2$, μ an \mathbb{F}_i -optional σ -integrable random measure on I . Then there exist \mathbb{F}_i -optional σ -integrable random measures μ^0, μ^c on I such that

$$(a) \quad \mu^0 = 1_{D^0(\mu) \times H} \mu, \text{ where}$$

$$D^0(\mu) = \{(\omega, t) \in \Omega \times I : \mu(\omega, \{t\} \times H) > 0\} \in \mathfrak{G}_i,$$

$$(b) \quad \mu^c = 1_{\overline{D^0(\mu)} \times H} \mu,$$

$$(c) \quad \mu = \mu^0 + \mu^c.$$

Moreover,

$$(d) \quad \mu^c \text{ is } \mathbb{F}_i\text{-previsible.}$$

If in addition, μ is \mathbb{F}_i -previsible, the same holds for its components, and $D^0(\mu) \in \mathfrak{P}_i$.

Proof. The previsibility of μ^c is a consequence of the continuity of associated integral processes, just as in the proof of Proposition 4. The rest of the assertion is a combination of Propositions 3 and 4. ■

As the final issue of this section, we will now single out a particularly important and simple class of random measures. They will be seen to contain the random measures associated with regular two-parameter martingales to be studied in the next section.

DEFINITION 6. 1. Let μ be a random measure on \mathbb{I} . Then μ is said to have “finite 0-jumps,” if $\mu(\omega, \{t\} \times H) < \infty$ for all $(\omega, t) \in \Omega \times \mathbb{I}$, “finite i -jumps,” if $\mu(\omega, \{t_i\} \times I \times H) < \infty$ for all $(\omega, t_i) \in \Omega \times I$, $i = 1, 2$. If μ has finite j -jumps for $j = 0, 1, 2$, μ is said to have “finite jumps.”

2. Let μ be a random measure on I . Then μ is said to have “finite jumps” if $\mu(\omega, \{t\} \times H) < \infty$ for all $(\omega, t) \in \Omega \times I$.

If in addition to being σ -integrable, a random measure has finite jumps, the sequence $(A_n)_{n \in \mathbb{N}}$ appearing in the definition of σ -integrability can be chosen in a particularly simple way.

PROPOSITION 5. Let μ be an optional σ -integrable random measure on \mathbb{I} with finite 0-jumps. Then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of 0-simple sets such that $T_n \uparrow D^0(\mu)$ and $1_{T_n \times H} \mu$ is integrable for $n \in \mathbb{N}$. A corresponding statement holds for random measures on I . If in addition, μ is i -previsible (previsible) σ -integrable, $(T_n)_{n \in \mathbb{N}}$ can be chosen i -previsible (previsible), $i = 1, 2$.

Proof. We prove the optional part of the assertion. Let $(A_n)_{n \in \mathbb{N}}$ and $(C_n)_{n \in \mathbb{N}}$ be as in the proof of Proposition 3. Set

$$T_n = \{(\omega, t) \in \Omega \times \mathbb{I} : A_t C_n(\omega) > 1/n, \mu(\omega, \{t\} \times H) < n\}, \quad n \in \mathbb{N}.$$

Then, since $\mu(\cdot, \{t\} \times H) = A_t(1_{\Omega \times \{t\} \times H} \cdot \mu)$ is optional, T_n is optional. It is 0-simple by [9, p. 105]. Moreover, by hypothesis, $T_n \uparrow D^0(\mu)$. The integrability of $1_{T_n \times H} \mu$ follows from the definition of simple sets and T_n , $n \in \mathbb{N}$. ■

PROPOSITION 6. *Let μ be an optional σ -integrable random measure on \mathbb{I} which satisfies $\mu(\omega, \{t\} \times H) = 0$ for $(\omega, t) \in \Omega \times \mathbb{I}$ and has finite j -jumps for $j = 1, 2$. Then there exists a sequence $(T_n^j)_{n \in \mathbb{N}}$ of j -simple sets such that $T_n^j \uparrow D^j(\mu)$ and $1_{T_n^j \times H} \mu$ is integrable for $n \in \mathbb{N}$. Moreover, $T_n^j \in \mathfrak{P}^j$, $n \in \mathbb{N}$.*

If in addition, μ is i -previsible (previsible) σ -integrable, $(T_n^j)_{n \in \mathbb{N}}$ can be chosen i -previsible (previsible), $i, j = 1, 2$.

Proof. With $(A_n)_{n \in \mathbb{N}}$, $(C_n)_{n \in \mathbb{N}}$ as in the proof of Proposition 3, we set

$$T_n^j = \{(\omega, t) \in \Omega \times \mathbb{I} : \Delta_{t_j}(C_n)_{(\cdot, t_j)}(\omega) > 1/n, \mu(\omega, \{t_j\} \times [0, t_j] \times H) < n\},$$

$n \in \mathbb{N}$, $j = 1, 2$. The optionality of T_n^j and its j -previsibility follow as before, using continuity of the defining processes in direction j . For j -simplicity cf. [9, p. 106]. ■

THEOREM 4. 1. *Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps, $j = 0, 1, 2$. Then there exists a sequence $(T_n^j)_{n \in \mathbb{N}}$ of j -simple sets such that*

$$(a) \quad T_n^0 \uparrow D^0(\mu), 1_{T_n^0 \times H} \mu \text{ is integrable,}$$

$$(b) \quad T_n^k \uparrow D^k(\mu'), 1_{T_n^k \times H} \mu \text{ is integrable, } k = 1, 2, \text{ where } \mu' = 1_{\overline{D^0(\mu)}} \times_H \mu$$

for all $n \in \mathbb{N}$. Moreover,

$$(c) \quad T_n^j \in \mathfrak{P}^j, n \in \mathbb{N}.$$

If in addition, μ is i -previsible (previsible) σ -integrable, the sequences can be chosen i -previsible (previsible), $i = 1, 2$.

2. *Let $i = 1, 2$, μ an \mathbb{F}_i -optional σ -integrable random measure on I with finite jumps. Then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of \mathbb{F}_i -simple sets such that*

$$T_n \uparrow D^0(\mu), 1_{T_n \times H} \mu \text{ is integrable for all } n \in \mathbb{N}.$$

If in addition, μ is \mathbb{F}_i -previsible σ -integrable, $(T_n)_{n \in \mathbb{N}}$ can be chosen \mathbb{F}_i -previsible.

Proof. Combine Propositions 5 and 6. ■

3. STOCHASTIC INTEGRALS OF POINT PROCESSES AND THE DECOMPOSITION OF SQUARE INTEGRABLE MARTINGALES

The principal aim of this section is to give a description of the jump components in the orthogonal decomposition of a square integrable two-parameter martingale in terms of stochastic integrals of point processes which are associated in a natural way with the jumps of the martingale. In the one-parameter case, this has been done, for example, in Jacod [10].

Given a square integrable regular martingale M , we associate with its 0-jumps the random measure (point process) μ^M which assigns unit mass to each point $(t, \Delta_t M)$ in $\mathbb{I} \times \mathbb{R}$ for which $\Delta_t M \neq 0$. If it is defined, the process $W_0 \cdot \mu^M$ describes the sum of all 0-jumps of M up to t , where $W_0(\cdot, \cdot, x) = x$, $x \in \mathbb{R}$. It is therefore clear that compensation of $W_0 \cdot \mu^M$ yields the 0-jump part M^0 of M . But, of course, W_0 need not be in $L^1(P \times \mu^M)$. For this reason, like Jacod [10], we will interpret the compensated process $W_0 \cdot \mu^M$ as a stochastic integral of W_0 with respect to the "compensation" $\mu^M - (\mu^M)^{\pi_1} - (\mu^M)^{\pi_2} + (\mu^M)^\pi$ of μ^M . As it stands, this expression does not necessarily make sense, since in general it does not represent a "signed" random measure, and has to be defined in an appropriate way. So our task can be put in the following terms: define the "compensated" stochastic integral $W^\varepsilon \cdot \mu^M$ for a sufficiently large class of previsible processes W and show that W_0 belongs to this class. In Theorem 5 the first main result is obtained: M^0 is identified with the compensated stochastic integral $W_0^\varepsilon \cdot \mu^M$. To describe the 1- and 2-jump parts M^1 and M^2 , we first consider the jumps of the one-parameter martingales $(M - M^0)_{(\cdot, 1)}$ and $(M - M^0)_{(1, \cdot)}$. We define a random measure (point process) $\mu_1^{M - M^0}$ on I which assigns unit point mass $(t_1, \Delta_{t_1}(M - M^0)_{(\cdot, 1)})$ in $I \times \mathbb{R}$ for which $\Delta_{t_1}(M - M^0)_{(\cdot, 1)} \neq 0$, and $\mu_2^{M - M^0}$, similarly, with respect to the jumps of $(M - M^0)_{(1, \cdot)}$. Now the compensated stochastic integrals $W^\varepsilon \cdot \mu_1^{M - M^0}$ and $W^\varepsilon \cdot \mu_2^{M - M^0}$ can be constructed in an analogous manner for previsible processes W on $\Omega \times I \times \mathbb{R}$, as in Jacod [10]. Taking $W_1^\varepsilon \cdot \mu_1^{M - M^0}$ and $W_1^\varepsilon \cdot \mu_2^{M - M^0}$, where $W_1(\cdot, \cdot, x) = x$, $x \in \mathbb{R}$, we obtain the jump parts of the two considered one-parameter martingales. But since for $t \in \mathbb{I}$ the processes $\Delta_{t_i}(M - M^0)_{(\cdot, 1)}$ are continuous \mathbb{F}_i -martingales, $i = 1, 2$, a simple optional projection of the stochastic integrals of W_1 in direction 2 (resp. 1) then yields M^1 and M^2 . In Theorem 6 this second main result will be established.

To start, we will define μ^M , μ_1^M , μ_2^M and show that they are random measures which fit into the framework of Section 1. Throughout this section, $H = \mathbb{R}$.

DEFINITION 7. Let M be a square integrable regular martingale. Then

$$\mu^M = \sum_{s \in \mathbb{I}} 1_{\{\Delta_s M \neq 0\}} \mathcal{E}_{(s, \Delta_s M)}$$

is called the "point process of 0-jumps" associated with M and

$$\mu_i^M = \sum_{s_i \in I} 1_{\{\Delta_{s_i} M_{(\cdot, 1)} \neq 0\}} \mathcal{E}_{(s_i, \Delta_{s_i} M_{(\cdot, 1)})},$$

the "point process of i -jumps" associated with M , $i = 1, 2$.

PROPOSITION 7. *Let M be a square integrable regular martingale. Then μ^M is an optional σ -integrable random measure on \mathbb{I} with finite jumps and $\mu^M = (\mu^M)^0$, μ_i^M an \mathbb{F}_i -optional σ -integrable random measure on I with finite jumps and $\mu_i^M = (\mu_i^M)^0$, $i = 1, 2$.*

Proof. We will establish the part of the assertion concerning μ^M . By definition, $\mu^M = (\mu^M)^0$ has finite jumps. By a standard argument, μ^M inherits the optionality of the process $\Delta.M$. For $n \in \mathbb{N}$ let

$$T_n = \{(\omega, t) \in \Omega \times \mathbb{I} : |\Delta_t M(\omega)| > 1/n\}, \quad A_n = T_n \cup \bigcup_{m \in \mathbb{N}} T_m \times \mathbb{R}.$$

Again by the optionality of $\Delta.M$, $T_n \in \mathfrak{G}$, hence $A_n \in \mathfrak{G} \times \mathfrak{B}(\mathbb{R})$, $n \in \mathbb{N}$. Now let $(\mathbb{J}_m)_{m \in \mathbb{N}}$ be a 0-sequence of partitions of \mathbb{I} . Then

$$\begin{aligned} E(1_{A_n} \cdot \mu_1^M) &\leq n^2 E \left(\sum_{(\cdot, s) \in T_n} (\Delta_s M)^2 \right) \\ &\leq n^2 E \left(\liminf_{m \rightarrow \infty} \sum_{J \in \mathbb{J}_m} (\Delta_J M)^2 \right) \\ &\leq n^2 \liminf_{m \rightarrow \infty} E \left(\sum_{J \in \mathbb{J}_m} (\Delta_J M)^2 \right) \quad (\text{Fatou}) \\ &= n^2 E(M_1^2) \quad (M \text{ is a martingale}). \end{aligned}$$

Since $A_n \uparrow \Omega \times \mathbb{I} \times \mathbb{R}$, μ^M is optionally σ -integrable and the proof is complete. ■

To define the stochastic integral with respect to μ^M or μ_i^M in the sense indicated above, in a first step we will consider those processes W , for which already $W \cdot \mu^M$ or $W \cdot \mu_i^M$ has a meaning. The following proposition then provides an idea for which class of previsible W the integral makes sense. It is stated for slightly more general random measures, after the introduction of an auxiliary notion for simple sets, connected with the distinction of previsible and inaccessible sets. For the same reasons as in the one-parameter theory, this distribution reveals its importance in any problem in which compensation plays a major role.

DEFINITION 8. 1. A 0-simple set T is called “pure” if

$$T = T^0 \cup T^1 \cup T^2 \cup T^c,$$

where T^0 is previsible, T^i is i -previsible, i -inaccessible, $i = 1, 2$, and T^c is totally inaccessible.

2. Let $i = 1, 2$. An \mathbb{F}_i -simple set T is called " \mathbb{F}_i -pure" if

$$T = T^0 \cup T^c,$$

where T^0 is \mathbb{F}_i -previsible and T^c is \mathbb{F}_i -inaccessible.

The system of pure (\mathbb{F}_i -pure) sets is denoted by \mathcal{P} (\mathcal{P}^i), $i = 1, 2$.

Remarks. 1. According to [9, pp. 110–116], the sets T^k , $k = 0, 1, 2, c$ (resp. $k = 0, c$) in the representation of a pure set T are pairwise disjoint and unique, up to evanescent sets.

2. Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps, $T \in D^0(\mu) \cup D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2}) \cup D^0(\mu^\pi)$ a pure set. Then

$$\begin{aligned} T^0 &= T \cap D^0(\mu^\pi), & T^i &= T \cap (D^0(\mu^{\pi_i}) \setminus D^0(\mu^\pi)), & i &= 1, 2, \\ T^c &= T \cap [D^0(\mu) \setminus D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2})]. \end{aligned}$$

A similar statement holds for random measures on I and \mathbb{F}_i -pure sets, $i = 1, 2$.

To see, for example, the last one of the stated equations, one has to prove that $S^c = T \cap [D^0(\mu) \setminus D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2})]$ is totally inaccessible. But $[I(S^c)^{\pi_i}]^0 = 0$, since $[(1_{\overline{D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2})} \times \mathbb{R}} \mu)^{\pi_i}]^0 = 0$, $i = 1, 2$, by definition and $S^c \subset D^0(\mu)$. Hence Proposition 2 of [9, p. 112] can be applied.

3. Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps. Then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of pure sets such that $T_n \uparrow D^0(\mu) \cup D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2}) \cup D^0(\mu^\pi)$ and $1_{T_n \times \mathbb{R}} \mu$ integrable for $n \in \mathbb{N}$. A similar statement holds for random measures on I . To see this, apply Theorem 4 to the random measures $\mu, \mu^{\pi_1}, \mu^{\pi_2}, \mu^\pi$, which is possible by Proposition 2, to choose sequences of 0-simple sets $(S_n^0)_{n \in \mathbb{N}}$ in \mathfrak{P} , $(S_n^i)_{n \in \mathbb{N}}$ in \mathfrak{P}^i , $i = 1, 2$, and $(S_n^c)_{n \in \mathbb{N}}$ in \mathfrak{G} such that $S_n^0 \uparrow D^0(\mu^\pi)$, $S_n^i \uparrow D^0(\mu^{\pi_i})$, $i = 1, 2$, $S_n^c \uparrow D^0(\mu)$. Now set

$$\begin{aligned} T_n &= S_n^0 \cup S_n^1 \cup S_n^2 \cup S_n^c, & T_n^0 &= S_n^0, & T_n^i &= S_n^i \setminus D^0(\mu^\pi), & i &= 1, 2, \\ T_n^c &= S_n^c \setminus D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2}), & n &\in \mathbb{N}, \end{aligned}$$

and look at Remark 2.

PROPOSITION 8. 1. Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps and such that $\mu = \mu^0$, $T \in \mathcal{P}^\sigma$. Then for $W \in \mathfrak{M}(\mathfrak{P} \times \mathfrak{B}(\mathbb{R}), \mathfrak{B}(\mathbb{R}))$ such that $W \in L^1(P \times (1_{T \times \mathbb{R}} \mu))$, the process

$$N^T = W \cdot (1_{T \times \mathbb{R}} \mu) - W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi_1} - W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi_2} + W \cdot (1_{T \times \mathbb{R}} \mu)^\pi$$

is a martingale and

$$\|N^T\|_{2,\infty} \leq 4 \left\| \sum_{(\cdot,t) \in T} (\Delta_t N^T)^2 \right\|_1^{1/2} \leq 4 \|N^T\|_{2,\infty}.$$

In particular, if $W \in L^1(P \times \mu)$,

$$\begin{aligned} \|N^T\|_{2,\infty} &\leq 4 \left\| \sum_{(\cdot,t) \in T} (\Delta_t W \cdot \mu - \Delta_t W \cdot \mu^{\pi_1} - \Delta_t W \cdot \mu^{\pi_2} + \Delta_t W \cdot \mu^\pi)^2 \right\|_1^{1/2} \\ &\leq 4 \|N^T\|_{2,\infty}. \end{aligned}$$

2. Let $i = 1, 2$, μ an \mathbb{F}_i -optional σ -integrable random measure on I with finite jumps and such that $\mu = \mu^0$, $T \in (\mathcal{P}^i)^\sigma$. Then for $W \in \mathfrak{M}(\mathfrak{P}_i \times \mathfrak{B}(\mathbb{R}), \mathfrak{B}(\mathbb{R}))$ such that $W \in L^1(P \times 1_{T \times \mathbb{R}} \mu)$, the process

$$N^T = W \cdot (1_{T \times \mathbb{R}} \mu) - W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi_i}$$

is an \mathbb{F}_i -martingale and

$$\|N^T\|_{2,\infty} \leq 2 \left\| \sum_{(\cdot,t) \in T} (\Delta_t N^T)^2 \right\|_1^{1/2} \leq 2 \|N^T\|_{2,\infty}.$$

In particular, if $W \in L^1(P \times \mu)$,

$$\|N^T\|_{2,\infty} \leq 2 \left\| \sum_{(\cdot,t) \in T} (\Delta_t W \cdot \mu - \Delta_t W \cdot \mu^{\pi_i})^2 \right\|_1^{1/2} \leq 2 \|N^T\|_{2,\infty}.$$

Proof. We prove the first part of the assertion. By Proposition 2,

$$W \in L^1(P \times (1_{T \times \mathbb{R}} \mu)^{\pi_i}), \quad i = 1, 2, \quad W \in L^1(P \times (1_{T \times \mathbb{R}} \mu)^\pi).$$

This means that N^T is well defined. If $F \times J$ is an i -previsible rectangle in $\Omega \times \mathbb{I}$, we have

$$\begin{aligned} E(1_{F \times J} W \cdot \mu_1^{\pi_i}) &= E(1_{F \times J} W \cdot \mu_1), \\ E(1_{F \times J} W \cdot \mu_1^\pi) &= E(1_{F \times J} W \cdot \mu_1^{\pi_i}), \quad i = 1, 2, \end{aligned}$$

due to Theorem 2. This implies that N^T is a martingale. Now let $(T_n)_{n \in \mathbb{N}}$ be a sequence of pure sets according to Remark 3 after Definition 8. For $n \in \mathbb{N}$, let

$$S_n = \left\{ (\omega, t) \in \Omega \times \mathbb{I} : \int_{\mathbb{R}} |W(\omega, t, x)| \mu(\omega, \{t\} \times dx) \leq n \right\}$$

and set

$$U_n = S_n \cap T_n \cap T.$$

Then U_n is also pure, since optional subsets of totally inaccessible sets are totally inaccessible. Omitting a standard completion argument, we may therefore assume that T is pure and that $W \cdot (1_{T \times \mathbb{R}} \mu)$ is of integrable variation. Moreover, we evidently may suppose that the expressions appearing in the asserted inequality are finite. Then, for any 0-sequence $(\mathbb{J}_m)_{m \in \mathbb{N}}$ of partitions of \mathbb{I} we obtain, using the martingale property of N^T and the purity of T

$$\begin{aligned}
\|N_1^T\|_2^2 &= \lim_{m \rightarrow \infty} \left\| \sum_{J \in \mathbb{J}_m} (\Delta_J N^T)^2 \right\|_1 \\
&\leq \left\| \limsup_{m \rightarrow \infty} \sum_{J \in \mathbb{J}_m} (\Delta_J N^T)^2 \right\|_1 \quad (\text{Fatou}) \\
&= \left\| \sum_{(\cdot, t) \in T^0} [\Delta_t W \cdot 1_{T \times \mathbb{R}} \mu - \Delta_t W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi_1} \right. \\
&\quad \left. - \Delta_t W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi_2} + \Delta_t W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi}]^2 \right. \\
&\quad \left. + \sum_{i=1,2} \sum_{(\cdot, t) \in T^i} [\Delta_t W \cdot 1_{T \times \mathbb{R}} \mu - \Delta_t W \cdot (1_{T \times \mathbb{R}} \mu)^{\pi_i}]^2 \right. \\
&\quad \left. + \sum_{(\cdot, t) \in T^c} [\Delta_t W \cdot 1_{T \times \mathbb{R}} \mu]^2 \right\|_1 \\
&\quad (\text{previsibility properties of } T^k, k=0, 1, 2, c) \\
&= \left\| \sum_{(\cdot, t) \in T} (\Delta_t N^T)^2 \right\|_1 \\
&= \left\| \liminf_{m \rightarrow \infty} \sum_{J \in \mathbb{J}_m} (\Delta_J N^T)^2 \right\|_1 \\
&\leq \lim_{m \rightarrow \infty} \left\| \sum_{J \in \mathbb{J}_m} (\Delta_J N^T)^2 \right\|_1 = \|N_1^T\|_2^2 \quad (\text{Fatou}).
\end{aligned}$$

Now it only remains to apply Doob's maximal inequality. ▀

Following Proposition 8, it is reasonable to define the stochastic integral with respect to μ for those processes W for which the square sums appearing in the inequalities are integrable.

DEFINITION 9. 1. Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps such that $\mu = \mu^0$. Then for $W \in \mathfrak{M}(\mathfrak{P} \times \mathfrak{B}(\mathbb{R}), \mathfrak{B}(\mathbb{R}))$ such that $W \in L^1(P \times \mu)^0$

$$\|W\|_\mu = \left\| \sum_{(\cdot, t) \in D^0(\mu) \cup \dots \cup D^0(\mu^\pi)} \left[\int_{\mathbb{R}} W(\cdot, t, x) (\mu(\cdot, \{t\} \times dx) - \mu^{\pi_1}(\cdot, \{t\} \times dx) - \mu^{\pi_2}(\cdot, \{t\} \times dx) + \mu^\pi(\cdot, \{t\} \times dx)) \right] \right\|_1^{1/2},$$

$$\mathfrak{H}(\mu) = \{W: W \in \mathfrak{M}(\mathfrak{P} \times \mathfrak{B}(\mathbb{R}), \mathfrak{B}(\mathbb{R})), W \in L^1(P \times \mu)^0, \|W\|_\mu < \infty\}.$$

2. Let $i = 1, 2$, μ an \mathbb{F}_i -optional σ -integrable random measure on I with finite jumps such that $\mu = \mu^0$. Then for $W \in \mathfrak{M}(\mathfrak{P}_i \times \mathfrak{B}(\mathbb{R}), \mathfrak{B}(\mathbb{R}))$ such that $W \in L^1(P \times \mu)^0$,

$$\|W\|_\mu = \left\| \sum_{(\cdot, t) \in D^0(\mu) \cup D^0(\mu^{\pi_i})} \left[\int_{\mathbb{R}} W(\cdot, t, x) \times (\mu(\cdot, \{t\} \times dx) - \mu^{\pi_i}(\cdot, \{t\} \times dx)) \right] \right\|_1^{1/2},$$

$$\mathfrak{H}(\mu) = \{W: W \in \mathfrak{M}(\mathfrak{P}_i \times \mathfrak{B}(\mathbb{R}), \mathfrak{B}(\mathbb{R})), W \in L^1(P \times \mu)^0, \|W\|_\mu < \infty\}.$$

Now we have to define the stochastic integral of $W \in \mathfrak{H}(\mu)$ w.r.t. μ . As usual, it could be conceivable to approximate, by choosing a sequence of “simple” functions $(W_n)_{n \in \mathbb{N}}$ in $\mathfrak{H}(\mu)$ converging to W , the integrals of which are given by Proposition 8. But, as we are about to see, it is more convenient to choose a sequence $(\mu_n)_{n \in \mathbb{N}}$ of “simple” random measures such that $W \in L^1(P \times \mu_n)$ for all n and which “converges” to μ in a reasonable sense.

PROPOSITION 9. *Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps and such that $\mu = \mu^0$, $W \in \mathfrak{H}(\mu)$. Then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of pure sets such that $T_n \uparrow D^0(\mu) \cup D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2}) \cup D^0(\mu^\pi)$ and such that for $\mu_n = 1_{T_n \times \mathbb{R}} \mu$, $\nu_n = 1_{\overline{T_n} \times \mathbb{R}} \mu$ we have*

- (a) μ_n is integrable,
- (b) $W \in L^1(P \times \mu_n)$, $n \in \mathbb{N}$,
- (c) $\|W\|_{\nu_n} \rightarrow 0$ ($n \rightarrow \infty$).

An analogous statement holds for random measures on I .

Proof. Choose $(S_n)_{n \in \mathbb{N}}$ according to Remark 3 after Definition 8 and for $n \in \mathbb{N}$ let

$$U_n = \left\{ (\omega, t) \in \Omega \times \mathbb{I} : \int_{\mathbb{R}} |W(\omega, t, x)| \mu(\omega, \{t\} \times dx) \leq n \right\}, \quad T_n = S_n \cap U_n.$$

Then T_n is pure. (a) follows by choice of T_n . Also, since S_n is 0-simple and

by definition of U_n , (b) follows. Finally, the purity of T_n and simple calculations yield

$$\begin{aligned} D^0(v_n) \cup D^0(v_n^{\pi_1}) \cup D^0(v_n^{\pi_2}) \cup D^0(v_n^\pi) \\ = D^0(\mu) \cup D^0(\mu^{\pi_1}) \cup D^0(\mu^{\pi_2}) \cup D^0(\mu^\pi) \setminus T_n, \end{aligned}$$

$n \in \mathbb{N}$. Hence

$$\begin{aligned} \|W\|_{v_n} = \left\| \sum_{(\cdot, t) \in D^0(\mu) \cup \dots \cup D^0(\mu^\pi) \setminus T_n} \left[\int_{\mathbb{R}} W(\cdot, t, x) \right. \right. \\ \left. \left. \times (\mu(\cdot, \{t\} \times dx) - \dots + \mu^\pi(\cdot, \{t\} \times dx)) \right] \right\|_1^{1/2} \\ \rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

by choice of $(T_n)_{n \in \mathbb{N}}$. This completes the proof. \blacksquare

PROPOSITION 10. *Let μ be an optional σ -integrable random measure on \mathbb{I} with finite jumps and such that $\mu = \mu^0$, $W \in \mathfrak{H}(\mu)$. Then there exists a unique square integrable martingale N such that for any sequence $(T_n)_{n \in \mathbb{N}}$ according to Proposition 9 the sequence $(N^{T_n})_{n \in \mathbb{N}}$ converges to N in \mathcal{M}^2 , where N^{T_n} is according to Proposition 8. Moreover,*

$$\|N\|_{2, \infty} \leq 4 \|W\|_\mu \leq 4 \|N\|_{2, \infty}.$$

An analogous statement holds for random measures on I .

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of pure sets according to Proposition 9 and define N^{T_n} as in Proposition 8, $n \in \mathbb{N}$. Then by Proposition 8 for $n, m \in \mathbb{N}$, $n \leq m$,

$$\|N^{T_m} - N^{T_n}\|_{2, \infty} = \|N^{T_m \setminus T_n}\|_{2, \infty} \leq 4 \|W\|_{1_{T_m \setminus T_n} \times \mathbb{R} \mu} \leq 4 \|W\|_{v_n},$$

where $v_n = 1_{\overline{T_n} \times \mathbb{R}} \mu$ as in the preceding proposition. Proposition 9(c) now yields the existence of N . The same argument also gives uniqueness, if one considers the union of two given sequences of pure sets. The inequality follows readily from the inequalities of Proposition 8. \blacksquare

DEFINITION 10. Let $i = 1, 2$, μ an (\mathbb{F}_i^-) optional σ -integrable random measure on \mathbb{I} (I) with finite jumps such that $\mu = \mu^0$, $W \in \mathfrak{H}(\mu)$. The martingale N according to Proposition 10 is called “stochastic integral of W w.r.t. μ ” and denoted by $W^c \mu$. (Sometimes, the adjective “compensated” is added.)

Now we can apply the general results just obtained to the particular case we are ultimately interested in, the jump components of a square integrable

martingale M . We have to verify that the process $W(\cdot, \cdot, x) = x$, $x \in \mathbb{R}$, is in $\mathfrak{H}(\mu^M)$ (resp. $\mathfrak{H}(\mu_i^M)$), $i = 1, 2$.

PROPOSITION 11. *Let M be a regular square integrable martingale, $W_0(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times \mathbb{I} \times \mathbb{R}$, $W_1(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times I \times \mathbb{R}$. Then $W_0 \in \mathfrak{H}(\mu^M)$, $W_1 \in \mathfrak{H}(\mu_i^M)$, $i = 1, 2$. Moreover,*

$$\begin{aligned} \|W_0 \circ \mu^M\|_{2,\infty} &\leq 4 \left\| \sum_{(\cdot, t) \in D^0(\mu^M)} (\Delta_t M)^2 \right\|_1^{1/2} \\ &\leq 4 \|W_0 \circ \mu^M\|_{2,\infty} \leq 4 \|M\|_{2,\infty}, \\ \|W_1 \circ \mu_i^M\|_{2,\infty} &\leq 2 \left\| \sum_{(\cdot, t_i) \in D^0(\mu_i^M)} (\Delta_{t_i} M_{(\cdot, 1)})^2 \right\|_1^{1/2} \\ &\leq 2 \|W_1 \circ \mu_i^M\|_{2,\infty} \leq 2 \|M\|_{2,\infty}, \quad i = 1, 2, \end{aligned}$$

and

$$\Delta \cdot M = \Delta \cdot W_0 \circ \mu^M, \quad \Delta \cdot M_{(\cdot, 1)} = \Delta \cdot W_1 \circ \mu_1^M, \quad \Delta \cdot M_{(1, \cdot)} = \Delta \cdot W_1 \circ \mu_2^M.$$

Proof. We concentrate on the two-parameter assertions. By definition, $W_0 \in L^1(P \times \mu^M)^0$. To compute $\|W_0\|_{\mu^M}$, let first

$$\begin{aligned} S_+^0 &= \left\{ (\omega, t) \in \Omega \times \mathbb{I} : \int_{\mathbb{R}} W_0(\cdot, t, x) (\mu^M)^\pi(\omega, \{t\} \times dx) > 0 \right\}, \\ S_-^0 &= \left\{ (\omega, t) \in \Omega \times \mathbb{I} : \int_{\mathbb{R}} W_0(\cdot, t, x) (\mu^M)^\pi(\omega, \{t\} \times dx) < 0 \right\} \end{aligned}$$

and S_+^i, S_-^i correspondingly with respect to $(\mu^M)^{\pi_i}$, $i = 1, 2$. Choose a sequence $(T_n^0)_{n \in \mathbb{N}}$ of 0-simple sets in \mathfrak{B} such that $T_n^0 \uparrow D^0((\mu^M)^\pi)$ and $1_{T_n^0 \times \mathbb{R}} \mu^M$ is integrable for all $n \in \mathbb{N}$. Assume, for example, that S_+^0 is non-evanescent. Since $S_+^0 \subset D^0((\mu^M)^\pi)$ and since both S_+^0 and T_n^0 are previsible, we obtain for n large enough

$$\begin{aligned} 0 &< E \left(\int_1 \int_{\mathbb{R}} W_0(\cdot, t, x) (\mu^M)^\pi(\cdot, \{t\} \times dx) d\Gamma(T_n^0 \cap S_+^0)_t \right) \\ &= E \left(\int_1 \int_{\mathbb{R}} W_0(\cdot, t, x) \mu^M(\cdot, \{t\} \times dx) d\Gamma(T_n^0 \cap S_+^0)_t \right) \quad (\text{Theorem 2}) \\ &= E \left(\int_1 \Delta_t M d\Gamma(T_n^0 \cap S_+^0)_t \right) \\ &= E \left(\int_1 {}^\pi(\Delta_t M) d\Gamma(T_n^0 \cap S_+^0)_t \right) = 0 \quad ({}^\pi \Delta \cdot M = 0, \text{ cf. [9, p. 87]}), \end{aligned}$$

a contradiction. Hence S_+^0 is evanescent. Analogously, S_-^0 , S_+^i , S_-^i are evanescent, $i = 1, 2$. This, however, implies

$$\|W_0\|_{\mu^M} = \left\| \sum_{(\cdot, t) \in D^0(\mu^M)} (\Delta_t M)^2 \right\|_1^{1/2}.$$

But as in the proof of Proposition 7,

$$\left\| \sum_{(\cdot, t) \in D^0(\mu^M)} (\Delta_t M)^2 \right\|_1 \leq \|M_1\|_2^2.$$

Therefore Proposition 10 yields the desired inequalities, hence the assertion. ■

We are ready to state our first main result.

THEOREM 5. *Let $W_0(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times \mathbb{I} \times \mathbb{R}$. Then the mapping*

$$K^0: \mathcal{M}^2 \rightarrow \mathcal{M}^2, \quad M \rightarrow W_0 \circ \mu^M$$

is an orthogonal projection. The orthogonal complement of $K^0(\mathcal{M}^2)$ is the space of all square integrable regular martingales without 0-jumps.

Proof. Using an approximation as in Proposition 10, it is easy to see that K^0 is a linear mapping. Proposition 11 implies its continuity. Now let $N = W_0 \circ \mu^M$. Then by Proposition 11,

$$W_0 \circ \mu^M = W_0 \circ \mu^N.$$

Hence K^0 is idempotent. Again by Proposition 11, the null space of K^0 is the space of all square integrable regular martingales without 0-jumps. But the latter space is also the orthogonal complement of the range of K^0 . This finally implies that K^0 is an orthogonal projection. ■

Given $M \in \mathcal{M}^2$, we next consider the orthogonal complement of $M^0 = W_0 \circ \mu^M$ w.r.t. M to describe its 1- and 2-jump components. To this end, we compensate the jumps of the martingales $(M - M^0)_{(\cdot, 1)}$ and $(M - M^0)_{(1, \cdot)}$ by taking the integrals $W_1 \circ \mu_1^{M - M^0}$ and $W_1 \circ \mu_2^{M - M^0}$ according to Proposition 11. As we will show in the following proposition, the optional projections of these processes in direction 2 (resp. 1) are the martingales we are looking for.

PROPOSITION 12. *Let M be a square integrable regular martingale*

without 0-jumps, $i = 1, 2$, $W_1(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times I \times \mathbb{R}$. Then the process

$${}^n[W_1 \circ \mu_i^M]$$

is a square integrable martingale without 0- and \bar{i} -jumps.

Proof. Let $i = 1$. Assume $(U_n)_{n \in \mathbb{N}}$ (resp. $(V_n)_{n \in \mathbb{N}}$) are sequences of previsible (resp. inaccessible) sets in $s(\mathcal{S}^1)$ such that the sequence $(T_n)_{n \in \mathbb{N}}$ of their unions is increasing and $\bigcup_{n \in \mathbb{N}} T_n \cap [\Omega \times [0, 1] \times \{1\}] \supset D^0(\mu_1^M)$ according to [9, p. 122]. As in [9, p. 118], consider the processes of 1-jumps of M on T_n ,

$$M(T_n) = M(U_n) + M(V_n), \quad n \in \mathbb{N}.$$

Theorem 3 of [9, p. 128], implies that they possess compensators C_n which are continuous and of bounded variation in direction 1. Set

$$N_n = M(T_n) - C_n, \quad n \in \mathbb{N}.$$

Then $(N_n)_{n \in \mathbb{N}}$ is a sequence of square integrable martingales without 0- and 2-jumps. We will show $N_n \rightarrow {}^n[W_1 \circ \mu_1^M]$ in \mathcal{M}^2 . This will imply the assertion. Let $p_i: \Omega \times \mathbb{I} \rightarrow \Omega \times [0, 1]$, $(\omega, t) \rightarrow (\omega, t_i)$, $i = 1, 2$. A simple geometrical consideration reveals that for $t_1 \in I$ the set $\overline{p_2(T_n) \cap [\Omega \times \{t_1\} \times [0, 1]]}$ consists of at most 3 (random) intervals, the length of which goes to zero as $n \rightarrow \infty$. Moreover, by choice of T_n , $D^0(\mu_1^M) \setminus p_1(T_n) \downarrow \emptyset$ as $n \rightarrow \infty$. Denote by $\mathcal{J}_n(t_1)$ the set of intervals of $\overline{p_2(T_n) \cap [\Omega \times \{t_1\} \times [0, 1]]}$ for $n \in \mathbb{N}$, $t_1 \in I$. As for Proposition 11 and by definition of $M(T_n)$, we obtain for $n \in \mathbb{N}$,

$$\|N_n - {}^n[W_1 \circ \mu_1^M]\|_{2, \infty} \leq 2 \|(N_n)_{(\cdot, 1)} - W_1 \circ \mu_1^M\|_{2, \infty} \quad (\text{Doob})$$

$$\begin{aligned} &\leq 4 \left\| \sum_{(\cdot, t_1) \in D^0(\mu_1^M) \setminus p_1(T_n)} (\Delta_{t_1} M_{(\cdot, 1)})^2 \right. \\ &\quad \left. + \sum_{(\cdot, t_1) \in p_1(T_n)} \sum_{J \in \mathcal{J}_n(t_1)} (\Delta_J \Delta_{t_1} M)^2 \right\|_1^{1/2}. \end{aligned}$$

Since M has no 0-jumps, the integrand in the last line of the preceding inequality converges to 0 pointwise as $n \rightarrow \infty$. We therefore have to find an integrable majorant in order to make Lebesgue's dominated convergence theorem finish the proof. Now

$$X = \sum_{(\cdot, t_1) \in D^0(\mu_1^M)} \sup_{t_2 \in I} (\Delta_{t_1} M_{(\cdot, t_2)})^2$$

is integrable, since for any 0-sequence $(\mathbb{J}_m)_{m \in \mathbb{N}}$ of partitions of I

$$X \leq \liminf_{m \rightarrow \infty} \sum_{J \in \mathbb{J}_m} \sup_{t_2 \in I} (\Delta_{J \times [0, t_2]} M)^2 = Y$$

and Y is integrable by Fatou's lemma and Doob's inequality. Finally, X dominates the integrand of the last line of the above inequality. This completes the proof. ■

The following theorem contains the second main result of this section.

THEOREM 6. *Let $W_1(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times I \times \mathbb{R}$, K^0 as in Theorem 5, $i = 1, 2$. Then the mapping*

$$K^i: K^0(\mathcal{M}^2)^\perp \rightarrow K^0(\mathcal{M}^2)^\perp, \quad M \rightarrow \gamma_i[W_1 \circ \mu_i^M],$$

is an orthogonal projection. K^0 , K^1 , K^2 are pairwise orthogonal. The orthogonal complement of $(K^0 + K^1 + K^2)(\mathcal{M}^2)$ is the space of all continuous square integrable martingales.

Proof. Proposition 12 shows that K^i is well defined. Linearity and continuity are handled as in the preceding proof. By Proposition 11, K^i is idempotent. By Propositions 11 and 12, the null space as well as the orthogonal complement of the range of K^i is the space of all martingales in \mathcal{M}^2 without 0- and \bar{i} -jumps. Hence K^i is an orthogonal projection, $K^1 K^2 = K^2 K^1 = 0$, and $(K^0 + K^1 + K^2)(\mathcal{M}^2)^\perp$ is the space of all martingales in \mathcal{M}^2 without 0-, 1- and 2-jumps. The assertion follows. ■

We finally summarize our main results in slightly different terms.

THEOREM 7. *Let $W_0(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times \mathbb{I} \times \mathbb{R}$ and $W_1(\omega, t, x) = x$ for $(\omega, t, x) \in \Omega \times I \times \mathbb{R}$. Every $M \in \mathcal{M}^2$ possesses the following orthogonal decomposition*

$$M = W_0 \circ \mu^M + \gamma_2[W_1 \circ \mu_1^{M-M^0}] + \gamma_1[W_1 \circ \mu_2^{M-M^0}] + M^c,$$

where $M^0 = W_0 \circ \mu^M$, $\gamma_i[W_1 \circ \mu_i^{M-M^0}]$ has no 0- and \bar{i} -jumps, $i = 1, 2$, M^c is continuous.

Proof. Combine Theorems 5 and 6. ■

Remark. The decomposition of Theorem 7 coincides with the decomposition of Theorem 1 of [9, p. 156].

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